Angular behavior of synchrotron radiation harmonics

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The detailed analysis of angular dependence of the synchrotron radiation (SR) is presented. Angular distributions of linear and circular polarization integrated over all harmonics, well known for relativistic electron energies, are extended to include radiation from electrons that are not fully relativistic. In particular, we analyze the angular dependence of the integral SR intensity and peculiarities of the angular dependence of the first harmonics SR. Studying spectral SR intensities, we have discovered their unexpected angular behavior, completely different from that of the integral SR intensity; namely, for any given synchrotron frequency, maxima of the spectral SR intensities recede from the orbit plane with increasing particle energy. Thus, in contrast with the integral SR intensity, the spectral ones have the tendency to deconcentrate themselves on the orbit plane.

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I. INTRODUCTION

At present the theory of synchrotron radiation (SR) is well developed and its predictions are in good agreement with experiment. General expressions for the spectral SR intensity (SR intensity for a fixed radiation frequency) were obtained in the framework of classical electrodynamics as far back as in Ref. [1], and since that time were represented in numerous books (e.g., Ref. [2], p. 20; Ref. [3], Sec. 10; Ref. [4], p. 31) and textbooks (e.g., Ref. [5], p. 676; Refs. [6], p. 296; [7,9], p. 21).

We recall that the SR is created by charged particles, which are moving with velocities v along circles of radius R in a uniform magnetic field H,

$$R=\frac{\beta E}{eH}=\frac{m_0c^2}{eH}\sqrt{\gamma^2-1},\quad \beta=\frac{v}{c},$$

$$\gamma=(1-\beta^2)^{-1/2}=\frac{E}{m_0c^2}>1. \tag{1}$$

Here E is the particle energy, e is the particle charge, and m_0 the rest mass of the particle. The radiation frequencies $\omega_\nu=\nu\omega_0, \nu=1,2,\cdots$, are multiples of the synchrotron frequency $\omega_0=ceH/E$. The spectral SR intensity (SR intensity for a fixed radiation frequency) has a maximum for harmonics with $\nu\sim\gamma^3$. Two limiting cases, the nonrelativistic ($\beta\ll 1, E\simeq m_0c^2$) and the relativistic limits ($\beta\sim 1, E\gg m_0c^2$), are of particular interest. In the nonrelativistic case, only the first harmonics $\omega_1=\omega$ is effectively emitted. The SR intensity has a maximum in the direction of the magnetic field. In the relativistic case, the integral SR intensity (spectral SR intensity summed over the spectrum) is concentrated in the orbital plane within a small interval $\Delta\theta\sim 1/\gamma\ll 1$ of the angle θ. We have chosen θ to be measured from the direction of the magnetic guide field which is normal to the orbital plane. Thus, as the electron energy increases, the integral SR intensity tends to be concentrated in the orbit plane. Any polarization component of the integral SR intensity has the same behavior. These results were first derived in the framework of classical theory. Consideration in the framework of quantum theory does not change essentially results of the classical analysis, since quantum corrections are small [2,3].

In this work we examine the angular dependence of polarization of the spectral and integral intensities of synchrotron radiation. Some years ago one of the authors, V.G.B., collaborated with the authors of Ref. [2] on their Chap. 1, where the present Eqs. (19) and (14) appeared. These equations give, respectively, the angular distance out of the orbital plane of the maximum emission of the component of synchrotron radiation that is linearly polarized orthogonal to the orbital plane, and the angular distances from the orbital plane of the two components of circularly polarized radiation, in the relativistic limit where β, the electron speed normalized to the speed of light, approaches the value of unity. In the present work this behavior is described for values of β intermediate between zero and unity. These results are sig-

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nificant for characterizing radiation by keV electrons in hot magnetized plasmas, for example.

In Sec. II we analyze in detail angular dependence of the integral SR intensity. In Sec. III we study peculiarities of the angular dependence of the first harmonics SR. Studying spectral SR intensities (see Sec. III), we have discovered their unexpected angular behavior, completely different from that of the integral SR intensity; namely, one can see that for any given synchrotron frequency, maxima of the spectral SR intensities recede from the orbit plane with increasing particle energy. There exist limiting angles (at $\beta\to 1$) for the maxima, which depend on the synchrotron frequency. Thus, in contrast with the integral SR intensity, the spectral ones have the tendency to deconcentrate themselves on the orbit plane. The analysis is done in the framework of classical theory, but as was already mentioned above, quantum corrections cannot change the results essentially. Below, some basic expressions (known from the above cited sources) are given; they are necessary for the presentation of the above results.

Using the notation of Ref. [2], we denote by index 1 the direction of propagation of the radiated photon, or the direction of the Poynting vector in the classical case. Indices 2,3 denote mutually orthogonal axes lying in a plane orthogonal to the propagation direction. Extremely relativistic electrons radiate synchrotron radiation mainly close to the direction of their instantaneous orbital velocity, so that the 3-axis can be chosen so as nearly to coincide with the direction of the magnetic guide field, traditionally called the π-direction, while the 2-axis will nearly coincide with the direction of centripetal acceleration, or σ-direction. Circular polarization vectors will lie along the positive and negative 1-axes, while transverse linear polarization components lie along the 2- and 3-axes. In the SR theory one introduces polarization components W_i, $i=0,\pm 1,2,3$ of the integral SR intensity. Here $W_{\pm 1}$ are the integral SR intensities of the right (+1) and the left (−1) circular polarization components, respectively, whereas W_2 and W_3 are the so called σ and π linear polarization components. The total integral SR intensity W_0 is defined as $W_0=W_1+W_{-1}=W_2+W_3$. In the framework of the classical theory of SR one can find

$$W_i = V_0\Phi_i(\beta), \quad V_0 = \frac{ce^2\beta^4}{R^2} = \frac{e^4H^2\beta^2(1-\beta^2)}{m_0^2c^3},$$

$$\Phi_i(\beta) = \int_0^\pi F_i(\beta,\theta)\sin\,\theta d\theta, \quad F_i(\beta,\theta) = \sum_{\nu=1}^{\infty} f_i(\nu,\beta;\theta),$$

$$f_0(\nu,\beta;\theta) = f_{-1}(\nu,\beta;\theta) + f_1(\nu,\beta;\theta) = f_2(\nu,\beta;\theta) + f_3(\nu,\beta;\theta). \tag{2}$$

Here θ is the angle between the z axis and the radiation direction. The particle orbit is placed in the plane $z=0$, which corresponds to $\theta=\pi/2$. In some works a different set of angles is used for the SR description. In particular, the z axis is selected to coincide with the direction of the particle's instantaneous velocity. The relation between the latter reference frame and the one used in the present paper is well known **[**see, e.g., Ref. [2], p. 35, Eqs. (6.6) and (6.7)**]**.

The sum over ν is just the sum over the spectrum, such that the expressions inside the sum represent spectral distributions. The functions $f_i(\nu,\beta;\theta)$ have the form

$$f_{\mp 1}(\nu,\beta;\theta) = \frac{\nu^2}{2}\left[J'_\nu(z) \mp \frac{\cos\,\theta}{\beta\sin\,\theta}J_\nu(z)\right]^2, \quad z=\nu\beta\sin\,\theta,$$

$$f_2(\nu,\beta;\theta) = \nu^2 J'^2_\nu(z), \quad f_3(\nu,\beta;\theta) = \frac{\nu^2\cos^2\,\theta}{\beta^2\sin^2\,\theta}J_\nu^2(z). \tag{3}$$

Here $J_\nu(x)$ are Bessel functions of integer indices. The following simple properties hold true:

$$f_k(\nu,\beta;\theta) = f_k(\nu,\beta;\pi-\theta), \quad k=0,2,3;$$

$$f_{-1}(\nu,\beta;\theta) = f_1(\nu,\beta;\pi-\theta). \tag{4}$$

Thus, it is enough to study the functions $f_k(\nu,\beta;\theta)$, $k=0,2,3$, at the interval $0\leqslant\theta\leqslant\pi/2$ only, and between the functions $f_{\pm 1}$ it is enough to study f_1 only.

Exact analytic expressions for the functions $F_k(\beta,\theta)$, $k=0,2,3$, have the following form [2–9]:

$$F_2(\beta,\theta) = \frac{7-3\varepsilon}{16\varepsilon^{5/2}}, \quad \varepsilon = 1-\beta^2\sin^2\theta, \quad \frac{1}{\gamma^2}\leqslant\varepsilon<1,$$

$$F_3(\beta,\theta) = \frac{(\gamma^2\varepsilon-1)(5-\varepsilon)}{16(\gamma^2-1)\varepsilon^{7/2}}, \tag{5}$$

$$F_0(\beta,\theta) = \frac{(3-4\gamma^2)\varepsilon^2 + 6(2\gamma^2-1)\varepsilon - 5}{16(\gamma^2-1)\varepsilon^{7/2}}.$$

Expressions for the functions $F_{\pm 1}$ can be found in the form

$$F_{\pm 1}(\beta,\theta) = \frac{1}{2}F_0(\beta,\theta) \pm \Psi(\beta\sin\,\theta)\cos\,\theta,$$

$$\Psi(x) = \frac{1}{2x}\frac{d}{dx}\sum_{\nu=1}^{\infty}\nu J_\nu^2(\nu x). \tag{6}$$

It remains the case that no closed analytic expression for the function $\Psi(x)$ has been found. We have therefore resorted to numerical computation where it occurs in our work. One can find by taking first and second derivatives that for any fixed β all the functions $F_i(\beta,\theta)$ have an extremum at $\theta=0$. Moreover, the extremal values of these functions do not depend on β,

$$F_{-1}(\beta,0) = 0, 2F_0(\beta,0) = 2F_1(\beta,0) = 4F_2(\beta,0) = 4F_3(\beta,0) = 1. \tag{7}$$

The point $\theta=\pi/2$ provides an extremum for the functions F_k, $k=0,2,3$ only. Here we have

$$F_0(\beta,\pi/2) = F_2(\beta,\pi/2) = 2F_{\pm 1}(\beta,\pi/2) = \frac{1}{16}\gamma^3(7\gamma^2-3),$$

$$F_3(\beta,\pi/2)=0. \tag{8}$$

Therefore, for F_3 the point $\theta=\pi/2$ is an absolute minimum. For any fixed β the function $F_2(\beta,\theta)$ is a monotonically increasing function of θ on the interval $0\leqslant\theta\leqslant\pi/2$. Thus, $\theta=0$ is an absolute minimum and $\theta=\pi/2$ is an absolute maximum of this function. The maximum of the function F_2 increases as E^5 with increasing particle energy E.

II. ANGULAR DEPENDENCE OF INTEGRAL SR INTENSITY

It is known that in the nonrelativistic case ($\beta\ll 1$) the point $\theta=0$ is a maximum for the functions F_0, F_1, and F_3. In the ultrarelativistic case ($\gamma\gg 1$) the maxima of these functions are shifted into the direction of the point $\theta=\pi/2$. However the behavior of these maxima in the region of varying normalized speed β and normalized energy γ of the radiating particle has not been analyzed before in detail. We will give such an analysis in what follows.

For the particle energy less than the lower critical value, $\gamma\leqslant\gamma_0^{(1)}$, ($\beta\leqslant\beta_0^{(1)}$), where

$$\gamma_0^{(1)}=\sqrt{7/6}\approx 1.0801, \quad \beta_0^{(1)}=1/\sqrt{7}\approx 0.378, \tag{9}$$

the integral intensity F_0 and circularly polarized components F_1 are monotonically decreasing functions of θ (F_0 on the interval $0\leqslant\theta\leqslant\pi/2$ and F_1 on the interval $0\leqslant\theta\leqslant\pi$). We recall that the function F_0 characterizes the integral intensity and the function F_1 characterizes the circularly polarized component of the intensity, see Eqs. (2). Thus, at $\theta=0$ these functions have an absolute maximum. The integral intensity F_0 and circularly polarized components F_1 have their absolute minima at $\theta=\pi/2$ and $\theta=\pi$, respectively. Besides, $F_1(\beta,\pi)=0$. In the particle energy interval between the two critical values $\gamma_0^{(1)}<\gamma<\gamma_0^{(2)}$, ($\beta_0^{(1)}<\beta<\beta_0^{(2)}$), where

$$\gamma_0^{(2)}=\frac{\sqrt{3}+3\sqrt{2}}{5}\approx 1.1949,$$

$$\beta_0^{(2)}=\sqrt{\frac{2}{3}(\sqrt{6}-2)}\approx 0.5474, \tag{10}$$

the points $\theta=0,\pi/2$ are minima for F_0, and the point $\theta=\theta_0(\beta)$,

$$\sin^2\theta_0(\beta)=\frac{6\gamma^2(1-3\gamma^2)+2\gamma^2\sqrt{15(15\gamma^4-22\gamma^2+9)}}{3(4\gamma^2-3)(\gamma^2-1)},$$

$$0<\theta_0(\beta)<\pi/2 \tag{11}$$

provides a maximum for F_0. For energies greater than the upper critical value $\gamma_0^{(2)}<\gamma$, ($\beta_0^{(2)}<\beta<1$), the function F_0 has an absolute maximum at the point $\theta=\pi/2$.

Denoting via $\theta_0^{(m)}(\beta)$ all the maximum points of F_0, we summarize as follows (see Fig. 1):

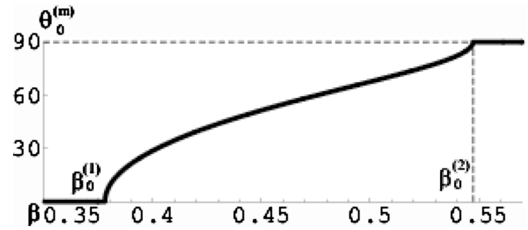

FIG. 1. The maximum points $\theta_0^{(m)}(\beta)$ of integral intensity F_0 plotted against normalized electron velocity β.

$$\theta_0^{(m)}(\beta)=\begin{cases}0, & \beta\leqslant\beta_0^{(1)}\\ \theta_0(\beta), & \beta_0^{(1)}<\beta<\beta_0^{(2)}\\ \pi/2, & \beta_0^{(2)}\leqslant\beta<1.\end{cases} \tag{12}$$

Turning to the integral intensity of the circular polarization, we find that for any given $\beta\in(\beta_0^{(1)},1)$, the function F_1 has its maximum at the point $\theta=\theta_1(\beta)$, $0<\theta_1(\beta)<\pi/2$. Denoting via $\theta_1^{(m)}(\beta)$ all the maximum points of F_1, we may write

$$\theta_1^{(m)}(\beta)=\begin{cases}0, & \beta\leqslant\beta_0^{(1)}\\ \theta_1(\beta), & \beta_0^{(1)}<\beta<1.\end{cases} \tag{13}$$

At the moment, there is no analytical expression for $\theta_1(\beta)$ similar to Eq. (11) for $\theta_0(\beta)$. However, one can see that the function $\theta_1(\beta)$ is a monotonically increasing function of $\beta\in[\beta_0^{(1)},1]$. In the limit of β approaching 1 we recover the asymptotic form that was given in Ref. [2],

$$\theta_1(\beta)\approx\pi/2-a_1/\gamma, \quad \beta\to 1, \tag{14}$$

where $a_1\approx 0.2672$ is a root of the equation (see Ref. [2])

$$5\pi a_1(5+12a_1^2)\sqrt{3}+64(5a_1^2-1)\sqrt{1+a_1^2}=0. \tag{15}$$

When we generalize to the region where β is increasing towards unity, we get the plot of the function $\theta_1^{(m)}(\beta)$ as shown in Fig. 2. In the case of linear polarization, where the normalized velocity β and energy of the radiating electron γ are less than the critical values β_3 and γ_3, respectively, where

$$\beta_3=\frac{2}{\sqrt{15}}\approx 0.5164, \quad \gamma_3=\sqrt{\frac{15}{11}}\approx 1.1678, \tag{16}$$

the integral intensity F_3 is a monotonically decreasing on the interval $0\leqslant\theta\leqslant\pi/2$ function. The point $\theta=0$ provides the absolute maximum for this function.

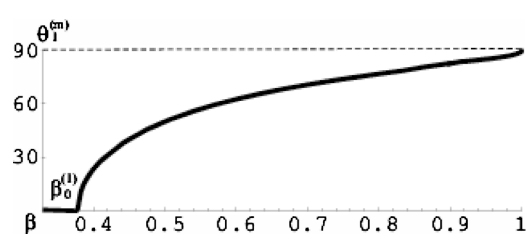

FIG. 2. The maximum points $\theta_1^{(m)}(\beta)$ of circularly polarized components F_1 plotted against normalized electron velocity β.

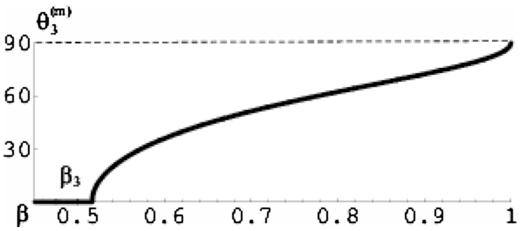

FIG. 3. The maximum points $\theta_3^{(m)}(\beta)$ of π linear polarization component F_3 plotted against normalized electron velocity β.

For values of normalized velocity β greater than the critical value β_3, that is, for $1>\beta>\beta_3$, $(\gamma>\gamma_3)$, the points $\theta=0$ and $\theta=\theta_3(\beta)$ provide the minimum and the maximum, respectively, for the integral intensity F_3,

$$\sin^2\theta_3(\beta)=\frac{\sqrt{5(125\gamma^4-34\gamma^2+5)}-19\gamma^2-5}{6(\gamma^2-1)},$$

$$0<\theta_3(\beta)<\pi/2. \qquad (17)$$

Denoting the angular position of all the maxima of the function F_3 by $\theta_3^{(m)}(\beta)$ (Fig. 3), we may write

$$\theta_3^{(m)}(\beta)=\begin{cases}0, & \beta\leqslant\beta_3\\ \theta_3(\beta), & \beta_3<\beta<1.\end{cases} \qquad (18)$$

For $\beta\to 1$ the following asymptotic expression previously given in Ref. [2] holds true:

$$\theta_3^{(m)}\approx\pi/2-\frac{1}{\gamma}\sqrt{\frac{2}{5}}. \qquad (19)$$

III. ANGULAR DEPENDENCE OF SPECTRAL SR INTENSITY

The behavior of some spectral components of SR intensity has been studied in the past by others both in the nonrelativistic and relativistic approximations. One can find the results of such an analysis in the above-cited literature. One ought to remark that the well-known relations in the ultrarelativistic case [10] are good approximations in the limits $\nu\sim\gamma^3$ and for $\theta\sim\pi/2$. For lower harmonics [5] a qualitative estimation $\theta\sim(3/\gamma)^{1/3}$ for the maxima is known. As will be seen below such an estimation holds true for $\nu\gg 1$ only. Moreover, we are going to demonstrate below that there exist several important peculiarities of the angular behavior of SR harmonics which are not described by the asymptotic formulas.

A. First harmonic radiation

The angular distribution of SR from the first harmonic $(\nu=1)$ is distinctly different from that of the higher harmonics $(\nu\geqslant 2)$. Previously it was known that (a) the first harmonic alone contributes essentially to the radiation in the directions $\theta=0,\pi$; (b) in the nonrelativistic case $(\beta\sim 0)$, the radiation is maximal exactly in these directions.

Let us consider Eqs. (3) for the first harmonic:

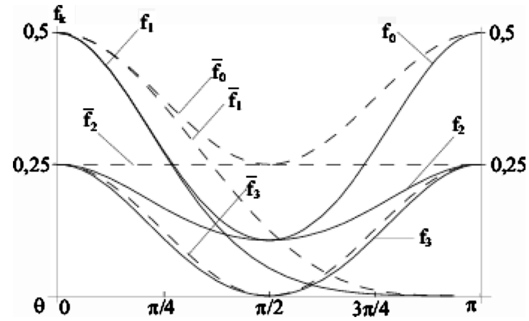

FIG. 4. The spectral intensity functions $\overline{f_k}=f_k(1,\beta=0;\theta)$ and $f_k=f_k(1,\beta=1;\theta)$ at two extremal values of electron velocity ($\beta=0$ and $\beta=1$) plotted against polar angle of emission θ (radians), for the different polarization components $k=0,1,2,3$ at harmonic number $\nu=1$.

$$f_{\mp 1}(1,\beta;\theta)=\frac{1}{2}\left[J_1'(z)\mp\frac{\cos\,\theta}{z}J_1(z)\right]^2,\quad z=\beta\sin\,\theta,$$

$$f_2(1,\beta;\theta)=J_1'^2(z),\qquad f_3(1,\beta;\theta)=\frac{\cos^2\,\theta}{z^2}J_1^2(z). \qquad (20)$$

In the nonrelativistic case $(\beta=0)$ we get

$$f_{\mp 1}(1,0;\theta)=\frac{1}{8}(1\mp\cos\,\theta)^2,\quad f_2(1,0;\theta)=\frac{1}{4},$$

$$f_3(1,0;\theta)=\frac{\cos^2\,\theta}{4}. \qquad (21)$$

Thus, in this case, the radiation components W_0 and W_3 peak at $\theta=0,\pi$, whereas W_2 does not depend on θ at all. Further analyzing the expressions (20), one can see that the functions $f_k(1,\beta;\theta)$, $k=0,1,2,3$, peak at $\theta=0$ for any β (including $\beta\to 1$). Thus, the corresponding radiation components W_k are maximal at $\theta=0$ for any β.

Besides, at any fixed $\theta\neq 0,\pi$, the functions $f_k(1,\beta;\theta),k=-1,0,1,2,3$, decrease monotonically with increasing β. Thus, the radiation from the first harmonic has the tendency to line up in the direction $\theta=0,\pi$ with increasing electron energy $(\beta\to 1)$. This behavior of the first harmonic radiation is completely opposite to that of the total SR intensity in the ultrarelativistic case. As was already said in the preceding section, the latter radiation tends to be concentrated in the orbital plane. All the functions given in Eq. (20) have finite limits as β approaches 1. Figure 4 presents the plots of particular cases of these functions, namely $\overline{f_k}$ for $\beta=0$, and f_k for $\beta=1$.

B. Higher harmonic radiation

To study the angular dependence of higher harmonic $(\nu>1)$ radiation, we have to analyze the angular dependence of the functions $f_k(\nu,\beta;\theta)$ for $\nu>1$.

First of all, one has to remark that all the functions $f_k(\nu,\beta;\theta)$, $\nu>1$ vanish at $\theta=0,\pi$. Thus, they have the mentioned absolute minima in directions parallel and antiparallel

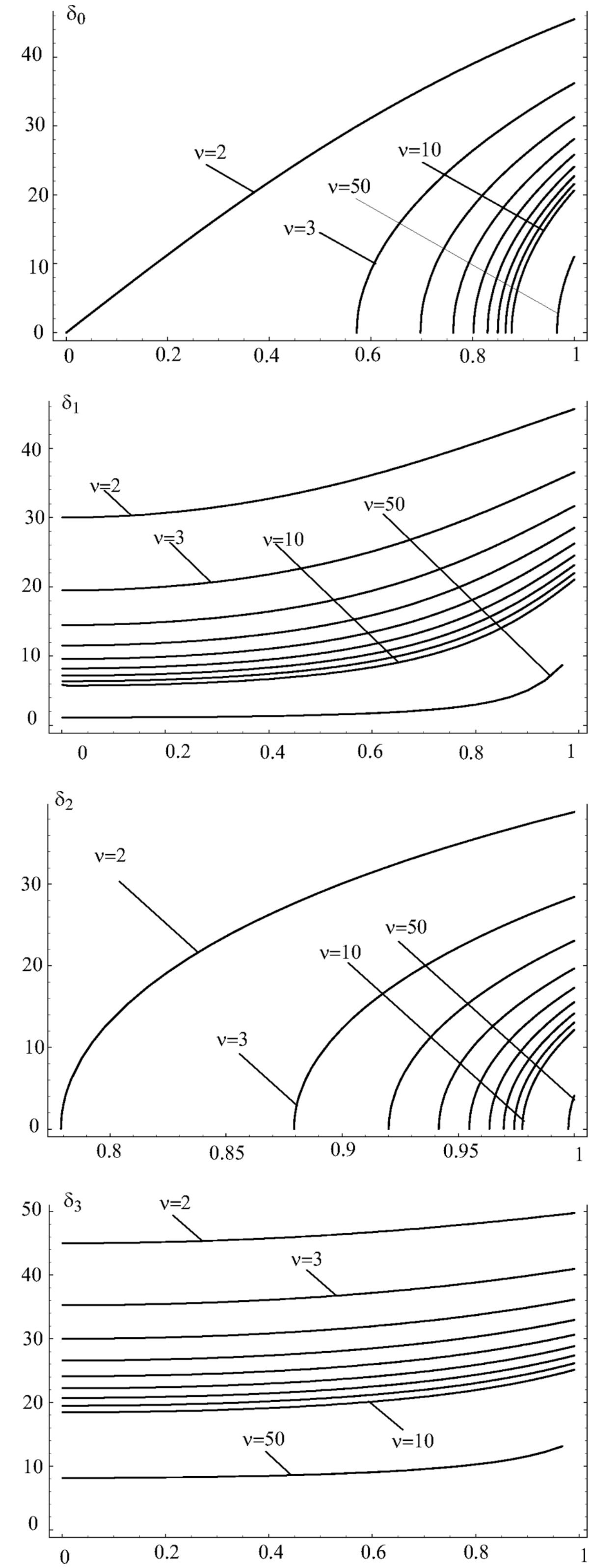

FIG. 5. Angles $\delta_k(\nu,\beta)$ of maximum emission plotted against normalized electron velocity β, for different polarization components $k=0,1,2,3$ at harmonic numbers $\nu=1,10,50$.

to the direction of the magnetic guide field. By virtue of Eq. (4) the functions $f_s(\nu,\beta;\theta), s=0,2,3$ have two symmetric maxima at the points

$$\theta_s^\nu(\beta) = \pi/2 \mp \delta_s(\nu,\beta), \quad s=0,2,3. \tag{22}$$

The functions $f_{\pm1}(\nu,\beta;\theta)$ have maxima at the points

$$\theta_{\pm1}^\nu(\beta) = \pi/2 \mp \delta_1(\nu,\beta). \tag{23}$$

The functions $\delta_k(\nu,\beta), k=0,1,2,3$, present deviations from the orbit plane of the SI-intensity maxima for each harmonic ν, for a given β, and for a given polarization component. All $\delta_k(\nu,\beta), k=0,1,2,3$, are increasing functions of β for any given ν, and for any given β they are decreasing functions of ν. At the same time,

$$0 \leqslant \delta_k(\nu,\beta) < \pi/2, \quad k=0,1,2,3 \tag{24}$$

and

$$\lim_{\beta\to1}\delta_k(\nu,\beta) = \delta_k(\nu,1) = \delta_k^\nu < \pi/2, \quad k=0,1,2,3. \tag{25}$$

The quantities δ_k^ν are maxima for $\delta_k(\nu,\beta)$ at fixed k,ν.

Thus, for each harmonic ($\nu>1$) and for each polarization component, the angular distribution of the SR intensity has its own maximum. For a given harmonic the angle of maximum emission of any polarization component recedes away from the orbit plane with increasing electron velocity, following a locus as shown in Fig. 5 that ends where $\beta=1$. The end point of the locus tends to lie closer to the orbital plane as the harmonic number increases. As the electron energy increases its radiation effectively is emitted by higher and higher harmonics, so that in the extreme relativistic limit the sum over all the harmonic intensities will closely resemble the integrated intensity which is concentrated at the orbital plane, as given by the well-known asymptotic formulas. Therefore, as in the case $\nu=1$, the spectral SR intensities for $\nu>1$ have the tendency to deconcentration from the orbit plane with increasing particle energy.

Below, we represent the behavior of all the functions $\delta_k(\nu,\beta)$. These results were obtained by applying both analytical and numerical methods.

We begin by considering the intensity polarized in the orbit plane. The extremum of the function $f_2(\nu,\beta;\theta)$ with respect to the argument θ is determined by the equation $df_2/d\theta=0$ which has the form

$$2J_v'(z)J_v''(z)\beta\cos(\theta)=0,$$

$$z=\nu\beta\sin(\theta)=\nu\beta\cos[\delta_2(\nu\beta)]<\nu.$$

Besides the obvious solution $\theta=\pi/2$ [that is, $\delta_2=0$, see Eq. (22)], this equation may have solutions that are defined by the following condition:

$$J_v''(z) = 0 = \left(\frac{\nu^2}{z^2}-1\right)J_\nu(z) - \frac{1}{z}J_\nu'(z). \tag{26}$$

Studying the latter condition (in particular, numerically for $\beta<\beta_2^\nu$), we may see that

$$\delta_2(\nu,\beta)=0, \quad \beta<\beta_2^\nu,$$

where β_2^ν is a root of the transcendental equation

$$\nu(1-\beta^2)J_\nu(\nu\beta)-\beta J_\nu'(\nu\beta)=0. \tag{27}$$

The function $J_\upsilon'(z)$ does not have any roots within the interval $0<z<\nu$.

Similar consideration applied to the function $f_0(\nu,\beta;\theta)$ leads us to the following result:

$$\delta_0(\nu,\beta)=0, \quad \beta<\beta_0^\nu,$$

where β_0^ν is a root of the transcendental equation

$$2\ J_\nu(\nu\ \beta)=\beta[\sqrt{\nu^2\ (1-\beta^2)^2-4}+\nu(1-\beta^2)]\ J_\nu'(\nu\ \beta). \tag{28}$$

The condition $\beta\leqslant 1$ implies that β_0^ν and β_2^ν are unique. The following inequality holds true:

$$\beta_0^\nu<\beta_2^\nu. \tag{29}$$

Since the Lorentz factor is by definition a function of β, see Eq. (1), the extremal values of β_k^ν correspond to the critical values of the Lorentz factor γ_k^ν for the k component of polarization,

$$\gamma_k^\nu=[1-(\beta_k^\nu)^2]^{-1/2}.$$

In particular, the critical values γ_0^ν and γ_2^ν of the Lorentz factor can be found from the asymptotic expressions at ν much greater than 1:

$$\gamma_0^\nu\approx\left(\frac{\nu}{a_0}\right)^{1/3}, \quad \gamma_2^\nu\approx\left(\frac{\nu}{a_2}\right)^{2/3}, \quad \nu\gg 1,$$

$$a_0=3z_0\approx 0.7332, \quad a_2=\left[\frac{16\pi^3}{\Gamma^6(1/3)\sqrt{3}}\right]^{1/4}\approx 0.9382. \tag{30}$$

Here z_0 is a root of the transcendental equation

$$3z_0K_{2/3}(z_0)-K_{1/3}(z_0)=0, \quad z_0\approx 0.2444,$$

where $K_\mu(x)$ are the Macdonald functions.

For $\beta>\beta_0^\nu$, the function $\delta_0(\nu,\beta)$ is defined as a solution of the transcendental equation

$$\begin{aligned}2\ J_\nu(\nu\beta\cos\ \delta_0)=\{&\nu[1-\beta^2+(1+\beta^2)\sin^2\ \delta_0]\\&+\sqrt{\nu^2[1-\beta^2+(1+\beta^2)\sin^2\ \delta_0]^2-4}\}\\&\times J_\nu'(\nu\beta\cos\ \delta_0)\beta\cos\ \delta_0.\end{aligned} \tag{31}$$

[One can see that Eq. (28) is a particular case of Eq. (31) at $\delta_0=0$.] Here $\delta_0(\nu,\beta)$ is a monotonically increasing function of β for each given ν. The maximum value δ_0^ν of $\delta_0(\nu,\beta)$ is a solution of Eq. (31) for $\beta=1$. There is the asymptotic (at $\nu\gg 1$) expression

$$\delta_0^\nu\approx\left(\frac{b_0}{\nu}\right)^{1/3}, \quad b_0=3p_0\approx 0.3066, \tag{32}$$

where p_0 is a root of the transcendental equation

$$6p_0K_{2/3}(p_0)-K_{1/3}(p_0)=0, \quad p_0\approx 0.1022. \tag{33}$$

The expression (32) coincides with the qualitative estimation presented in Ref. [5]. The latter fact indicates that such an estimation holds true for $\nu\gg 1$ only.

For $\beta>\beta_2^\nu$, the function $\delta_2(\nu,\beta)$ has the form

$$\delta_2(\nu,\beta)=\arccos(\beta_2^\nu/\beta), \quad \beta\in(\beta_2^\nu,1). \tag{34}$$

Therefore, the maximum value δ_2^ν of $\delta_2(\nu,\beta)$ at the point $\beta=1$ is

$$\delta_2^\nu=\delta_2(\nu,1)=\arccos\ \beta_2^\nu. \tag{35}$$

According to Eq. (30), we have the asymptotic (at $\nu\gg 1$) expression

$$\delta_2^\nu\approx 1/\gamma_2^\nu, \tag{36}$$

where the emission angle δ_2^ν is expressed in radians and γ_2^ν is the critical value of the Lorentz factor for the 2-component of polarization.

Turning to the 1- and 3-components of polarization, the functions $\delta_1(\nu,\beta)$ and $\delta_3(\nu,\beta)$ behave similar to $\delta_0(\nu,\beta)$ and $\delta_2(\nu,\beta)$; namely, $\delta_1(\nu,\beta)$ and $\delta_3(\nu,\beta)$ are defined as solutions of the transcendental equations

$$\begin{aligned}(\nu\sin\ \delta_1-1-\nu\beta^2\sin\ \delta_1\cos^2\ \delta_1)J_\nu(\nu\beta\cos\ \delta_1)\\=J_\upsilon'(\nu\beta\cos\ \delta_1)\beta(1-\nu\sin\ \delta_1)\sin\ \delta_1\cos\ \delta_1\end{aligned} \tag{37}$$

and

$$J_\upsilon'(\nu\beta\cos\ \delta_3)\nu\beta\sin^2\ \delta_3\cos\ \delta_3=J_\nu(\nu\beta\cos\ \delta_3), \tag{38}$$

respectively. For each given ν, these functions are bounded and monotonically increasing functions of $\beta\in[0,1]$,

$$\arcsin(1/\nu)=\delta_1(\nu,0)\leqslant\delta_1(\nu,\beta)\leqslant\delta_1^\nu=\delta_1(\nu,1),$$

$$\arcsin(1/\sqrt{\nu})=\delta_3(\nu,0)\leqslant\delta_3(\nu,\beta)\leqslant\delta_3^\nu=\delta_3(\nu,1). \tag{39}$$

For each given β, these functions decrease monotonically with increasing ν.

At $\nu\gg 1$ we get the following asymptotic expressions:

$$\delta_1^\nu\approx\left(\frac{b_1}{\nu}\right)^{1/3}, \quad b_1=3p_1\approx 0.3933, \quad \delta_3^\nu\approx\left(\frac{a_0}{\nu}\right)^{1/3}, \tag{40}$$

where a_0 is defined by Eq. (30), and $p_1\approx 0.13114$ is a root of the transcendental equation

$$(3p_1-1)K_{1/3}(p_1)+3p_1K_{2/3}(p_1)=0. \tag{41}$$

The following inequalities hold true:

$$\delta_3(\nu,\beta)>\delta_1(\nu,\beta)>\delta_0(\nu,\beta)\geqslant\delta_2(\nu,\beta). \tag{42}$$

The threshold values γ_0^ν, γ_2^ν and the extremal values δ_ν^k for some ν are given in Tables I and II.

TABLE I. The critical values γ_0^ν, γ_2^ν of the Lorentz factor.

ν	2	3	4	5	6	7	10	15	20	25	30	35	40	45	50	100	200	300	400	500
γ_0^ν	1.00	1.22	1.40	1.54	1.67	1.79	2.08	2.46	2.75	3.00	3.21	3.40	3.58	3.74	3.88	4.98	6.35	7.31	8.07	8.70
γ_2^ν	1.59	2.10	2.55	2.97	3.36	3.73	4.75	6.25	7.59	8.82	9.98	11.07	12.10	13.10	14.06	22.38	35.58	46.66	56.54	65.63

Figure 5 shows how the polarization maxima move away from the orbital plane for all harmonics as the velocity of the radiating electron increases towards $\beta=1$. These loci become shorter as the harmonic number increases, confining the maxima closer to the orbital plane.

Note in Fig. 6 the unexpected small dip in the 2-component of polarization at an angle of $\pi/2$, corresponding to a direction lying in the orbital plane, which is present even at the highest harmonic number of 500 that we have calculated. This minimum has remained unsuspected until now, the asymptotic solutions having failed to reveal it. The maxima lie ever closer to the orbital plane as the harmonic number increases. At ultrarelativistic electron energies, very high harmonics make the most significant contribution to the observed intensity. Thus in the high energy limit the polarization maxima will approach each other very closely from opposite sides of the orbital plane, in agreement with the well-known result [5,8] that the integral intensity has its maximum in the orbital plane at high energy. However this dip should be measurable at electron energies lower than 10 MeV where the critical harmonic, and hence the harmonic number of any intense frequency component, is in the vicinity of the 500th harmonic which is the highest harmonic that we have listed in Table II.

Integrating a spectral SR intensity of k-polarization component over all the directions, one can obtain the so called total spectral SR intensity of k-polarization component. Denoting via $\nu_k^{\max}(\beta)$ the harmonic for which the latter quantity is maximal, and considering β dependence of this harmonic, one can see that $\nu_k^{\max}(\beta)$ is a step function which breaks in curve at the points $\beta_k(n)$, $n=1,2,\cdots$, and is constant on each interval $[\beta_k(n),\beta_k(n+1)]$. In the paper [11], positions of the points $\beta_k(n)$ were studied in detail. One can use these results to get additional information about the behavior of the functions $\delta_0(\nu,\beta)$ and $\delta_2(\nu,\beta)$. In particular, one can derive that the function $\delta_0(\nu,\beta)$ is not zero for $\nu=\nu_0^{\max}(\beta)$, whereas the function $\delta_2(\nu,\beta)$ equals zero for $\nu=\nu_2^{\max}(\beta)$.

IV. SUMMARY

We have analyzed the angular behavior of SR harmonics in the framework of classical theory. The main physical deductions are the following.

It was demonstrated that for any given harmonic the maximum of the radiation does not have any tendency to concentrate itself on the orbit plane with increasing particle energy. On the contrary, in this case there is a completely opposite behavior: the maximum recedes from the orbit plane. Moreover, with infinite increase of the particle energy the maximum of the radiation for each harmonic tends to its own finite limiting value, which characterizes the harmonic

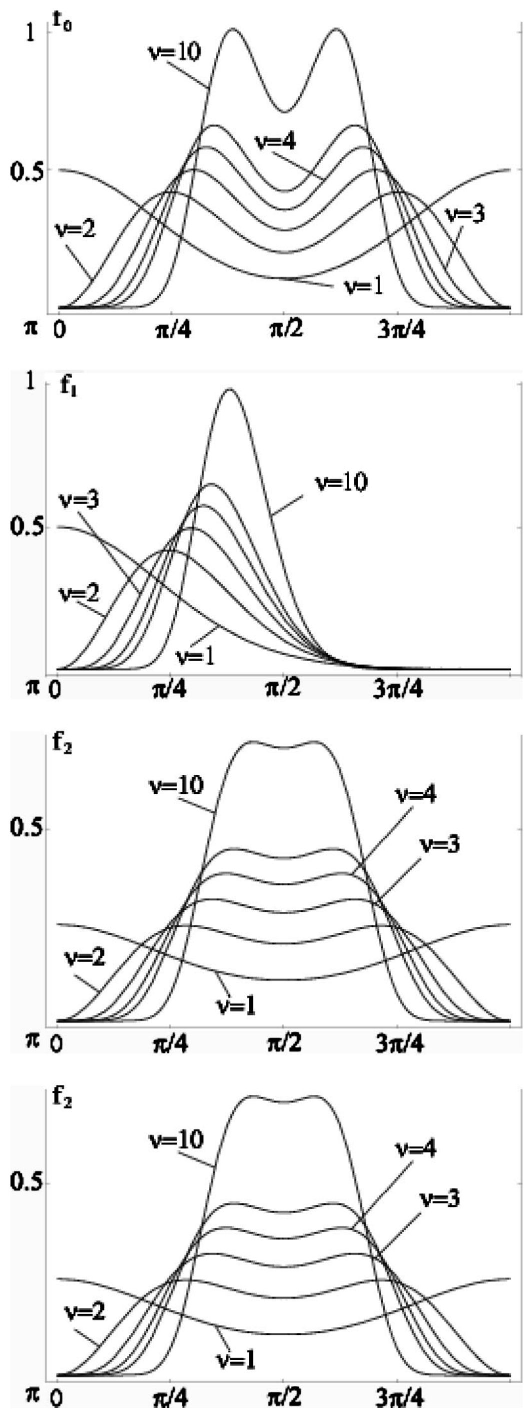

FIG. 6. Spectral intensity functions $f_k(\nu,1;\theta)$ at fixed electron velocity ($\beta=1$) of the different polarization components $k=0,1,2,3$ for harmonic numbers $\nu=1,2,3,4,5,10$ plotted against polar angle of emission θ (radians) relative to the direction of the magnetic guide field.

TABLE II. The extremal values δ_ν^k of maximum emission angles at fixed k,ν (in deg).

ν	2	3	4	5	6	7	10	15	20	25	30	35	40	45	50	100	200	300	400	500
δ_0^ν	45.50	36.22	31.29	28.11	25.84	24.10	20.66	17.48	15.59	14.30	12.34	12.59	11.98	11.47	11.03	8.60	6.74	5.86	5.31	4.92
δ_1^ν	45.88	36.83	32.02	28.91	26.68	24.98	21.57	18.39	16.48	15.16	14.18	13.41	12.77	12.24	11.79	9.24	7.28	6.34	5.75	5.33
δ_2^ν	38.84	28.44	23.06	19.67	17.30	15.54	12.14	9.20	7.57	6.51	5.75	5.18	4.74	4.38	4.08	2.56	1.61	1.23	1.01	0.87
δ_3^ν	49.83	41.09	36.29	33.11	30.80	29.00	25.34	21.83	19.69	18.19	17.06	16.16	15.43	14.81	14.28	11.26	8.90	7.76	7.04	6.53

deconcentration. At a fixed particle energy this deconcentration decreases monotonically with increasing the harmonic number. This statement holds true for any polarization component.

Results obtained here can be considered to be applied in space [12] and laboratory [13] plasmas. Especially these effects can be important for runaway electron discharges [12], where electron velocities are relativistic or nonrelativistic and electron radiation lies mainly in the electron cyclotron frequency's region. Results of the present paper can be also applied for elaborating diagnostics of electron cyclotron emission radiometry type, where detailed experimental analysis of frequencies of particle radiation and its angular distribution can be performed. Of course, in plasmas collective effects are strong, and the problem of using of present results can be considered in future investigations. Nevertheless, the substantial deviation of the maximum angle of the synchrotron radiation from the electron orbital plane, analogous to the result obtained in the present paper, has been observed in tokamak plasmas at some special conditions (see, e.g., Ref. [14]). To perform detailed comparison of these experimental results with our predictions is a goal of future investigations.

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